The zero density theorem for the Rankin-Selberg L-function and its applications

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Abstract

In this work, we establish a zero density result for the Rankin-Selberg L-functions. As an application, we apply it to distinguish the holomorphic Hecke eigenforms for $\mathrm{SL}_2(\mathbb{Z})$.

1 Introduction

The study of zero free region of L-functions is a classical theme in analytic number theory. It is widely believed that the generalized Riemann hypothesis (GRH) holds, but a proof for this is out of research at present. In the absence of GRH, the zero density estimates, especially the number of zeros near the line $\mathrm{Re}(s) = 1$, are often used as a substitute in many applications. Such type results are often referred as the *zero density theorems*. For the zero density theorem of Riemann zeta function $\zeta(s)$, one can refer to [Mon71, Chapter 12]. For the zero density theorem for Dirichlet L-functions $L(s, \chi)$, one can refer to [Mon69] and [IK04, Theorem 1.4].

Later the zero density theorem for automorphic L-functions was well studied in different aspects. In [Luo99], Luo established a zero density result for the symmetric square L-function of Maass forms with large eigenvalues. Later Kowalski and Michel [KM02] proved a general density theorem for automorphic L-functions with large conductors. In their 2006 work, Lau and Wu [LW06] established a zero density theorem for the holomorphic cusp forms in the weight aspect. Recently Thorner and Zaman [TZ21] proved a zero density result for automorphic L-functions over a number field F by first establishing an unconditional large sieve inequality for automophic forms on $\mathrm{GL}_n(\mathbb{A}_F)$. For the applications of the zero density theorems, one can also refer to the papers above.

In this paper, we will study the zero density results for the Rankin-Selberg L-function of $\mathrm{SL}_2(\mathbb{Z})$ holomorphic Hecke eigenforms. This will extend the zero free region for Rankin-Selberg functions in the weight aspect *averagely*. As a direct application, we can combine it with [GH93] to distinguish Hecke eigenforms.

We proceed to our results. Let k be an even number. Denote by $\mathcal{S}_k(\mathrm{SL}_2(\mathbb{Z}))$ the (finite dimensional) vector space of weight k holomorphic cusp forms for $\mathrm{SL}_2(\mathbb{Z})$. Then it has an orthogonal basis with respect to the Petersson inner product, denoted by H_k, which are also eigenfunctions for

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Hecke operators. Let $f \in H_k$ be a Hecke eigenform. Then it has a Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{\frac{k-1}{2}} e(nz)$$

and we can assume that $\lambda_f(1) = 1$. The associated L-function, denoted by $L(s, f)$, has a Euler product:

$$L(s, f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p \left(1 - \frac{\alpha_p}{p^s}\right)^{-1} \left(1 - \frac{\alpha_p^{-1}}{p^s}\right)^{-1}.$$

Here $\{\alpha_p, \alpha_p^{-1}\}$ are the Satake parameters of f at p. The well-known result by Deligne is $|\alpha_p| = |\alpha_p^{-1}| = 1$ for all p.

Now let g be another Hecke eigenform in H_k with Satake parameters $\{\beta_p, \beta_p^{-1}\}$ for all primes p. Then the Rankin-Selberg L-function, denoted by $L(s, f \otimes g)$, is defined by

$$L(s, f \otimes g) = \prod_p \left(1 - \frac{\alpha_p \beta_p}{p^s}\right)^{-1} \left(1 - \frac{\alpha_p \beta_p^{-1}}{p^s}\right)^{-1} \left(1 - \frac{\alpha_p^{-1} \beta_p}{p^s}\right)^{-1} \left(1 - \frac{\alpha_p^{-1} \beta_p^{-1}}{p^s}\right)^{-1}$$

when $\mathrm{Re}(s) > 1$. The Rankin-Selberg L-function has a meromorphic continuation to the whole complex plane. It can only has a simple pole at $s = 1$ and this happens if and only if $f = g$. In this case, $L(s, f \otimes f) = \zeta(s) L(s, \mathrm{Sym}^2 f)$ with

$$L(s, \mathrm{Sym}^2 f) = \prod_p \left(1 - \frac{\alpha_p^2}{p^s}\right)^{-1} \left(1 - \frac{1}{p^s}\right)^{-1} \left(1 - \frac{\alpha_p^{-2}}{p^s}\right)^{-1}$$

when $\mathrm{Re}(s) > 1$. Here $L(s, \mathrm{Sym}^2 f)$ is the symmetric square L-function assocaited to f. It has an analytic continuation to the whole complex plane.

Next we would introduce a function counting the number of zeros. Let $L(s, \pi)$ be an L-function defined in [IK04, Chapter 5]. (π later will be taken to be $\mathrm{Sym}^2 f$ or $f \otimes g$.) For fixed $\alpha > \frac{1}{2}$, denote by $N(\alpha, T, \pi)$ the number of zeros $\rho = \beta + i\gamma$ of $L(s, \pi)$ with $\beta \geq \alpha$ and $0 \leq \gamma \leq T$. We can establish the following zero density results for the Rankin-Selberg L-function:

Theorem 1.1. *Denote by $\mathcal{F}_k$ the set of pairs of distinct Hecke eigenfucntions, that is,*

$$\mathcal{F}_k = \{(f, g) \in H_k \times H_k | f \neq g\},$$

For any $\delta > 0$, we have

$$\sum_{(f,g) \in \mathcal{F}_k} N(\alpha, T, f \otimes g) \ll_\delta T^2 (\log T) k^{34(1-\alpha)/(3-2\alpha)} (\log k)^{25}$$

uniformly for $\frac{1}{2} + \delta \leq \alpha \leq 1$, $(\log k)^3 \leq T \leq k$ and $f, g \in H_k$. The constant is dependent on δ when δ is small. When $\frac{1}{2} + \delta$ is closed to 1, the implied constant is absolute.

In a recent paper [BTZ22], Brumley, Thorner and Zaman established a zero density theorem of the Rankin-Selberg L-functions under certain Hypothesis. In their paper, they fixed one cuspidal representation π_0 and varied the other representation in a given family. In our work, we can vary two modular forms (see the definition of $\mathcal{F}_k$). This can be achieved by Lemma 2.1. To prove the theorem, we will first establish a large sieve inequality and then follow the work in [LW06, Section 5].

For each $\eta \in (0, 1/2)$, define

$$H_k^+(\eta) = \{f \in H_k | L(s, \mathrm{Sym}^2 f) \neq 0 \text{ for } s \in \mathcal{S}\}$$

and

$$D_k^+(\eta) = \{(f, g) \in \mathcal{F}_k | L(s, f \otimes g) \neq 0 \text{ for } s \in \mathcal{S}\},$$

where $S = \{s | \operatorname{Re}(s) \geq 1 - \eta, | \operatorname{Im}(s) \leq 100k^\eta|\} \cup \{s | \operatorname{Re}(s) \geq 1\}$.

Then set $H_k^-(\eta) = H_k - H_k^+(\eta)$ and $D_k^-(\eta) = \mathcal{F}_k - D_k^+(\eta)$. The following corollary shows that $D_k^+(\eta)$ has density one as $k \to \infty$ provided that η is small:

Corollary 1.2. *For $\eta \in \left(\frac{3 \log \log k}{\log k}, \frac{1}{4}\right)$, we have*

$$|D_k^-(\eta)| \ll k^{36\eta} (\log k)^{26}$$

The implied constant is absolute.

Proof: This is similar to [LW06, Equation 1.11]. When $\eta < \frac{1}{4}$, $\delta \geq \frac{1}{4}$. The implied constant is absolute due to Remark 3.2. □

Remark 1.3. *A similar argument with Remark 3.3 shows that*

$$|H_k^-(\eta)| \ll k^{36\eta} (\log k)^{18}.$$

Next we will consider an application of the zero density theorem: let $f, g \in H_k$ be distinct Hecke eigenforms. We say that f and g are *distinguishable* if for every $\epsilon > 0$, we can find $n \ll_\epsilon k^\epsilon$ such that $\lambda_f(n) \neq \lambda_g(n)$. Combine Corollary 1.2 and Remark 1.3, and we establish the following result:

Theorem 1.4. *As $k \to \infty$, we can find a set of Hecke eigenforms H_k^* ($\subseteq H_k$) such that*

$$\lim_{k \to \infty} \frac{|H_k^*|}{|H_k|} = 1$$

and for any $f, g \in H_k^*$ and $f \neq g$, they are distinguishable.

The proof is based on [GH93].

2 The large sieve inequality

In this section, we will prove the following large sieve inequality, which will be used to prove Theorem 1.1 in the next section. Before that, we need the following lemma to know the possible poles for Rankin-Selberg L-functions:

Lemma 2.1. *Let $f_1, f_2, g_1, g_2 \in H_k$ be Hecke eigenforms. Assume that $f_1 \neq g_1$ and $f_2 \neq g_2$. Then $L(s, f_1 \otimes g_1 \otimes f_2 \otimes g_2)$ has at most a simple pole at $s = 1$. This happens if and only one of the following is valid:*

(a) $f_1 = f_2$ and $g_1 = g_2$;

(b) $f_1 = g_2$ *and* $f_2 = g_1$.

Proof. ($\Rightarrow$) By the condition that $f_1 \neq g_1$ and $f_2 \neq g_2$, we know that the $L(s, f_1 \otimes g_1 \otimes f_2 \otimes g_2)$ will have a pole at $s = 1$ if one of (a), (b) holds.

($\Leftarrow$) If exactly two of f_1, f_2, g_1, g_2 are equal, then $L(s, f_1 \otimes g_1 \otimes f_2 \otimes g_2)$ is entire. Without loss of generality, we assume that $f_1 = f_2$ and f_1, g_1, g_2 are distinct. Then

$$L(s, f_1 \otimes g_1 \otimes f_2 \otimes g_2) = L(g_1 \otimes g_2)L(s, \mathrm{Sym}^2 f_1 \otimes g_1 \otimes g_2)$$

is entire due to [Ram00] and [CKM04, Theorem 9.2]. The left cases can be discussed similarly. So it suffices to consider the case when any two of f_1, f_2, g_1, g_2 are distinct. Then by [Ram00], we know that, for each pair of distinct $f, g \in \{f_1, f_2, g_1, g_2\}$, there exists an automorphic cuspidal representations of $\mathrm{GL}_4(\mathbb{A}_\mathbb{Q})$, denoted by $\Pi_{f,g}$, such that $L(s, f \otimes g) = L(s, \Pi_{f,g})$. Then by [CKM04, Theorem 9.2], $L(s, f_1 \otimes g_1 \otimes f_2 \otimes g_2)$ will have at most a simple pole at $s = 1$. We prove the rest by contradiction: suppose that $L(s, f_1 \otimes g_1 \otimes f_2 \otimes g_2)$ has a simple pole at $s = 1$. Then for any $x \geq 1$, we can show that

$$\sum_{p \leq x} \lambda_{f_1}(p)\lambda_{f_2}(p)\lambda_{g_1}(p)\lambda_{g_2}(p) = \frac{x}{\log x} + O_A\left(\frac{x}{\log^A x}\right) \tag{1}$$

by [IK04, Theorem 5.13]. Here $A > 0$ is arbitrary. (Notice that f_1, f_2, g_1, g_2 satisfy the Ranmanujan conjecture and hence [IK04, Equation (5.48)] is obvious.)

On the other hand, [CKM04, Theorem 9.2] implies that

$$\Pi_{f_1,f_2} \cong \Pi_{g_1,g_2} \qquad \Pi_{f_1,g_1} \cong \Pi_{f_2,g_2} \qquad \Pi_{f_1,g_2} \cong \Pi_{f_2,g_1}.$$

Therefore, for each prime p, we have

$$\lambda_{f_1}(p)\lambda_{f_2}(p) = \lambda_{g_1}(p)\lambda_{g_2}(p) \qquad \lambda_{f_1}(p)\lambda_{g_1}(p) = \lambda_{f_2}(p)\lambda_{g_2}(p) \qquad \lambda_{f_1}(p)\lambda_{g_2}(p) = \lambda_{f_2}(p)\lambda_{g_1}(p).$$

Multiply them together, and we obtain that

$$\lambda_{f_1}(p)^3\lambda_{f_2}(p)\lambda_{g_1}(p)\lambda_{g_2}(p) = \lambda_{f_2}(p)^2\lambda_{g_1}(p)^2\lambda_{g_2}(p)^2.$$

Let S be a set of primes satisfying

$$S = \{p | \lambda_{f_2}(p)\lambda_{g_1}(p)\lambda_{g_2}(p) = 0\}.$$

Since $L(s, f_1 \otimes f_1 \otimes f_1 \otimes f_1)$ has a pole of order 2 at $s = 1$, we have

$$\begin{aligned}\sum_{p \leq x} \lambda_{f_1}(p)\lambda_{f_2}(p)\lambda_{g_1}(p)\lambda_{g_2} &= \sum_{p \leq x, p \notin S} \lambda_{f_1}(p)\lambda_{f_2}(p)\lambda_{g_1}(p)\lambda_{g_2}(p) \\ &\geq \sum_{p \leq x} \lambda_{f_1}(p)^4 - 16 \sum_{p \leq x, p \in S} 1 \qquad (|\lambda_{f_1}(p)| \leq 2) \\ &= \frac{2x}{\log x} + O_A\left(\frac{x}{\log^A x}\right) - 16 \sum_{p \leq x, p \in S} 1\end{aligned} \tag{2}$$

However $p \in S$ implies that one of $\lambda_{f_2}(p), \lambda_{g_1}(p), \lambda_{g_2}(p)$ is zero. This cannot happen quite frequently due to the Sato-Tate conjecture [BLGHT11]. ($\lambda_f(p) = 0$ implies that one of Satake parameters α_p, α_p^{-1} is $e^{i\pi/2}$) Indeed, we can show that for any $\epsilon > 0$,

$$\sum_{p \leq x, p \in S} 1 \leq \frac{\epsilon x}{\log x}$$

as $x \to \infty$. Then combine this with Equation (1) and (2), a contradiction. $\square$

Then we prove the following large sieve inequality for the family $\mathcal{F}_k$:

Lemma 2.2. *Let $L \geq 1$. Let $\{a_\ell\}_{\ell \leq L}$ be a sequence of complex numbers. Then for any $\epsilon > 0$, we have*

$$\sum_{(f,g)\in\mathcal{F}_k} \left| \sum_{\ell \leq L} a_\ell \lambda_{f\otimes g}(\ell) \right|^2 \ll_\epsilon (L(\log k)^{15} + k^{9/2+\epsilon} L^{1/2+\epsilon}) \sum_{\ell\leq L} |a_\ell|^2.$$

Proof By duality principal, it suffices to show:

$$\sum_{\ell \leq L} \left| \sum_{(f,g)\in\mathcal{F}_k} b_{f,g} \lambda_{f\otimes g}(\ell) \right|^2 \ll_\epsilon (L\log L(\log k)^{15} + k^{9/2+\epsilon} L^{1/2+\epsilon}) \sum_{f,g\in H_k} |b_{f,g}|^2.$$

The left hand side is

$$\ll \sum_{\ell \geq 1} \left| \sum_{(f,g)\in\mathcal{F}_k} b_{f,g} \lambda_{f\otimes g}(\ell) \right|^2 e^{-\ell/L} = \sum_{\substack{(f_1,g_1)\in\mathcal{F}_k \\ (f_2,g_2)\in\mathcal{F}_k}} b_{f_1,g_1} \overline{b_{f_2,g_2}} \sum_{\ell\geq 1} \lambda_{f_1\otimes g_1}(\ell)\lambda_{f_2\otimes g_2}(\ell) e^{-\ell/L}. \quad (3)$$

A standard argument will show that

$$\begin{aligned}\sum_{\ell\geq 1} \lambda_{f_1\otimes g_1}(\ell)\lambda_{f_2\otimes g_2}(\ell) e^{-\ell/L} &= \frac{1}{2\pi i}\int_{(2)} L(s, f_1\otimes g_1\otimes f_2\otimes g_2) G_{f_1,f_2,g_1,g_2}(s)\Gamma(s)L^s\, ds \\ &= \mathrm{Res}_{s=1} L(s, f_1\otimes g_1\otimes f_2\otimes g_2) G_{f_1,f_2,g_1,g_2}(s)\Gamma(s)L^s \\ &\quad + \frac{1}{2\pi i}\int_{(1/2+\epsilon)} L(s, f_1\otimes g_1\otimes f_2\otimes g_2) G_{f_1,f_2,g_1,g_2}(s)\Gamma(s)L^s\, ds\end{aligned}$$

Here $G_{f_1,f_2,g_1,g_2}(s)$ is some Euler product which is absolutely convergent for $\mathrm{Re}(s) > 1/2$. Additionally, we have $G_{f_1,f_2,g_1,g_2}(s) \ll_\epsilon 1$ for $\mathrm{Re}(s) \geq \frac{1}{2} + \epsilon$ (independent from the choice of f_1, f_2, g_1 and g_2 since they satisfy the Ranmanujan conjecture.) By the virtual of the proof bellow, it suffices to assume that $G_{f_1,f_2,g_1,g_2}(1) \neq 0$.

For the function $L(s, f_1\otimes g_1\otimes f_2\otimes g_2)$, it will have a pole of order 1 if and only if either $f_1 = f_2 \neq g_1 = g_2$ or $f_1 = g_2 \neq f_2 = g_1$ by Lemma 2.1. In both cases, we have

$$L(s, f_1\otimes g_1\otimes f_2\otimes g_2) = \zeta(s)L(s, \mathrm{Sym}^2 f_1)L(s, \mathrm{Sym}^2 g_1)L(s, \mathrm{Sym}^2 f_1\otimes \mathrm{Sym}^2 g_1).$$

Since f_1, f_2, g_1, g_2 satisfies the Ranmanujan conjecture, we can show:

$$L(1, \mathrm{Sym}^2 f) \ll (\log k)^3$$

and

$$L(1, \mathrm{Sym}^2 f\otimes \mathrm{Sym}^2 g) \ll (\log k)^9$$

by [CM04, Lemma 4.1]. This will show that

$$\mathrm{Res}_{s=1} L(s, f_1\otimes g_1\otimes f_2\otimes g_2) G_{f_1,f_2,g_1,g_2}(s)\Gamma(s)L^s \ll (\delta_{f_1,f_2}\delta_{g_1,g_2} + \delta_{f_1,g_2}\delta_{f_2,g_1}) L(\log k)^{15}.$$

Here $\delta_{f,g} = 1$ if $f = g$ and 0 otherwise. By [LW06, Section 3], one can show that

$$L(1/2+\epsilon+it, f_1 \otimes g_1 \otimes f_2 \otimes g_2) \ll_\epsilon (1+|t|)^{6/4}(k+|t|)^{5/2+\epsilon}. \tag{4}$$

This will show that

$$\sum_{\ell\geq 1} \lambda_{f\otimes g_1}(\ell)\lambda_{f\otimes g_2}(\ell)e^{-\ell/L} \ll_\epsilon (\delta_{f_1,f_2}\delta_{g_1,g_2} + \delta_{f_1,g_2}\delta_{f_2,g_1})L(\log k)^{15} + k^{5/2+\epsilon}L^{1/2+\epsilon}. \tag{5}$$

Notice that, by Cauchy's inequality,

$$\sum_{(f,g)\in\mathcal{F}_k} |b_{f,g}\overline{b_{g,f}}| \leq \left(\sum_{(f,g)\in\mathcal{F}_k} |b_{f,g}|^2\right).$$

Insert Equation (5) to Equation (3) and we obtain the result. □

Remark 2.3. *By a similar argument, one can show that the term Lk^ϵ in [LW06, Proposition 4.1] can be replaced by $L(\log k)^8$.*

3 Proof of Theorem 1.1

Our proof is based on the method of Montgomery in [Mon71]. Here we will follow the work in [LW06, Section 5]. We will modify their work to replace k^ϵ by a large power of $\log k$. Indeed, this also works for the symmetric square L-functions and we will discuss it in Remark 3.3.

We first establish the following lemma, which is similar to [LW06, Lemma 5.1]:

Lemma 3.1. *Let $z > 16$ be any fixed number and let $P(z) = \prod_{p\leq z} p$. For any $\mathrm{Re}(s) = \sigma > 1$, we have*

$$L(s, f\otimes g)^{-1} = G_{f,g}(s) \sum_{(n,P(z))=1} \frac{\lambda_{f\otimes g}(n)\mu(n)}{n^s}.$$

The Dirichlet series $G_{f,g}(s)$ converges absolutely for $\sigma > \frac{1}{2}$ and $G_{f,g}(s) \ll_\epsilon 1$ uniformly for $\mathrm{Re}(s) \geq \frac{1}{2}+\epsilon$. Notice that the implied constant is independent from the choice of f, g.

Then for $\mathrm{Re}(s) > 1/2$, we define

$$M_x(s, f\otimes g) = G_{f,g}(s) \sum_{\ell\leq x, (\ell,P(z))=1} \frac{\mu(\ell)\lambda_{f\otimes g}(\ell)}{\ell^s}.$$

The trivial bound shows that, for any $\epsilon > 0$

$$M_x(s, f\otimes g) \ll_\epsilon x^{1/2}$$

uniformly for $\mathrm{Re}(s) \geq \frac{1}{2}+\epsilon$. Obviously, we have

$$1 = (1 - L(s, f\otimes g)M_x(s, f\otimes g)) + L(s, f\otimes g)M_x(s, f\otimes g).$$

Proof of Theorem 1.1: We cut the rectangle $\alpha \leq \mathrm{Re}(s) \leq 1$ and $0 \leq \mathrm{Im}(s) \leq T$ horizontally into boxes of width $2(\log k)^2$. Then by [IK04, Propositon 5.7], each box $\alpha \leq \mathrm{Re}(s) \leq 1$ and $Y \leq \mathrm{Im}(s) \leq Y + 2(\log k)^2$ contains at most $O((\log k)^3)$ zeros. Denote by $n_{f\otimes g}$ the number of boxes which contains at least a zero of $L(s, f\otimes g)$. Then

$$N(\alpha, T, f\otimes g) \ll n_{f\otimes g}(\log k)^3 \ll n_{f\otimes g}T.$$

So to prove Theorem 1.1, it suffices to show: for a fixed $\alpha \geq \frac{1}{2}+\delta$,

$$\sum_{(f,g)\in\mathcal{F}_k} n_{f\otimes g} \ll_\delta T(\log T)k^{34(1-\alpha)/(3-2\alpha)}(\log k)^{25}.$$

Let x, y be large numbers to be chosen later. (We will choose x, y to be some power of k and such choice is sufficient for the following argument.) Let $\rho = \beta + i\gamma$ with $\beta \geq \alpha (> \frac{1}{2}+\epsilon)$ and $0 \leq \gamma \leq T$ be a zero of $L(s, f\otimes g)$ and we write:

$$\kappa = \frac{1}{\log k} \qquad \kappa_1 = 1 - \beta + \kappa (>0) \qquad \kappa_2 = \frac{1}{2} - \beta + \epsilon (<0).$$

(Here it suffices to assume that $\epsilon < \delta$.) Then follow [LW06, Section 5], we have:

$$\begin{aligned} e^{-1/y} = &\frac{1}{2\pi i}\int_{(\kappa_1)} (1 - L(\rho+\omega, f\otimes g)M_x(\rho+\omega, f\otimes g))\Gamma(\omega)y^\omega \, d\omega \\ &+ \frac{1}{2\pi i}\int_{(\kappa_2)} L(\rho+\omega, f\otimes g)M_x(\rho+\omega, f\otimes g)\Gamma(\omega)y^\omega \, d\omega \end{aligned}$$

By the convexity bound in [LW06, Proposition 3.1]:

$$L(s, f\otimes g) \ll_{\epsilon'} (1+|\mathrm{Im}(s)|)^{1-\mathrm{Re}(s)}(k+|\mathrm{Im}(s)|)^{1-\mathrm{Re}(s)+\epsilon'} \tag{6}$$

for any $\epsilon' > 0$, we have:

$$\int\limits_{(\kappa_1), |\mathrm{Im}(\omega)|\geq(\log k)^2} (1 - L(\rho+\omega, f\otimes g)M_x(\rho+\omega, f\otimes g))\Gamma(\omega)y^\omega \, d\omega \ll_\epsilon \frac{1}{k^2}$$

and

$$\int\limits_{(\kappa_2), |\mathrm{Im}(\omega)|\geq(\log k)^2} L(\rho+\omega, f\otimes g)M_x(\rho+\omega, f\otimes g)\Gamma(\omega)y^\omega \, d\omega \ll_\epsilon \frac{1}{k^2}.$$

This is due to the rapid decay of $\Gamma(\omega)$ when $|\mathrm{Im}(\omega)| \geq (\log k)^2$. In this case, set $K = (\log k)^2$ and by the fact that $1 \leq C(a+b) \Rightarrow 1 \leq 2C^2(a^2+b)$ (where $a, b > 0$ and $C \geq 1$,), we obtain:

$$\begin{aligned} 1 \ll (\log k)^2 y^{2-2\alpha} &\int_{-K}^{K} |1 - L(1+\kappa+i(\gamma+v), f\otimes g)M_x(1+\kappa+i(\gamma+v), f\otimes g)|^2 \, dv \\ + y^{1/2-\alpha+\epsilon} &\int_{-K}^{K} |L(1/2+\epsilon+i(\gamma+v), f\otimes g)M_x(1/2+\epsilon+i(\gamma+v), f\otimes g)| \, dv. \end{aligned}$$

Then follow the work in [LW06, Page 457], and we have:

$$\begin{aligned}
n_{f\otimes g} &\ll (\log k)^2 y^{2-2\alpha} \int_0^{2T} |1 - L(1+\kappa+iv, f\otimes g) M_x(1+\kappa+iv, f\otimes g)|^2 \, dv \\
&\quad + y^{1/2-\alpha+\epsilon} \int_0^{2T} |L(1/2+\epsilon+iv, f\otimes g) M_x(1/2+\epsilon+iv, f\otimes g)| \, dv \\
&=: (\log k)^2 y^{2-2\alpha} \mathrm{I}_{f\otimes g} + y^{1/2-\alpha+\epsilon} \mathrm{II}_{f\otimes g}.
\end{aligned}$$

By Equation (6), we have, for any $\epsilon' > 0$,

$$L(1/2+\epsilon+iv, f\otimes g) \ll_{\epsilon'} (1+|v|)^{1/2}(k+|v|)^{1/2+\epsilon'} \ll_{\epsilon'} k^{1+\epsilon'}$$

provided that $T \leq k$. Therefore, for $\mathrm{II}_{f\otimes g}$, we have:

$$\mathrm{II}_{f\otimes g} \ll_{\epsilon,\epsilon'} T x^{1/2} k^{1+\epsilon'}. \tag{7}$$

Then set $X = e^{4(\log k)^2}$ and we have

$$\begin{aligned}
1 - &L(1+\kappa+iv, f\otimes g) M_x(1+\kappa+iv, f\otimes g) \\
&= L(1+\kappa+iv, f\otimes g) G_{f,g}(1+\kappa+iv) \sum_{\ell > x, (\ell, P(z))=1} \frac{\mu(\ell)\lambda_{f\otimes g}(\ell)}{\ell^{1+\kappa+iv}} \\
&\ll_\epsilon \zeta(1+\kappa)^4 \left(\left| \sum_{x<\ell\leq X, (\ell,P(z))=1} \frac{\mu(\ell)\lambda_{f\otimes g}(\ell)}{\ell^{1+\kappa+iv}} \right| + \sum_{\ell > X} \frac{d_4(\ell)}{\ell^{1+\kappa}} \right) \\
&\ll_\epsilon \zeta(1+\kappa)^4 \left| \sum_{x<\ell\leq X, (\ell,P(z))=1} \frac{\mu(\ell)\lambda_{f\otimes g}(\ell)}{\ell^{1+\kappa+iv}} \right| + X^{-\kappa/2} \zeta(1+\kappa/2)^4 \zeta(1+\kappa)^4
\end{aligned}$$

Recall that $\kappa = \frac{1}{\log k}$ and this will give:

$$\sum_{(f,g)\in\mathcal{F}_k} \mathrm{I}_{f\otimes g} \ll_\epsilon (\log k)^4 \int_0^{2T} \sum_{(f,g)\in\mathcal{F}_k} \left| \sum_{x<\ell\leq X, (\ell,P(z))=1} \frac{\mu(\ell)\lambda_{f\otimes g}(\ell)}{\ell^{1+\kappa+iv}} \right|^2 dv + T(\log k)^8.$$

Then by dyadic division, Lemma 2.2 and Cauchy-Schwarz inequality, we obtain:

$$\sum_{(f,g)\in\mathcal{F}_k} \mathrm{I}_{f\otimes g} \ll_{\epsilon,\epsilon_1} (\log k)^4 (\log X)^2 T \left((\log k)^{15} + k^{9/2+\epsilon_1} x^{-1/2+\epsilon_1} \right). \tag{8}$$

Combine Equation (8) and Equation (7), and we have:

$$\begin{aligned}
\sum_{(f,g)\in\mathcal{F}_k} n_{f\otimes g} &\ll_\epsilon (\log k)^2 y^{2-2\alpha} \sum_{(f,g)\in\mathcal{F}_k} \mathrm{I}_{f\otimes g} + y^{1/2-\alpha+\epsilon} \sum_{(f,g)\in\mathcal{F}_k} \mathrm{II}_{f\otimes g} \\
&\ll_{\epsilon,\epsilon_1,\epsilon'} (\log k)^2 y^{2-2\alpha} \left(T(\log k)^{23} + T k^{9/2+\epsilon_1} x^{-1/2+\epsilon_1} (\log k)^4 \right) + y^{1/2-\alpha+\epsilon} T x^{1/2} k^{3+\epsilon'}
\end{aligned}$$

Set $\epsilon_1 = \frac{1}{22}, \epsilon' = \frac{1}{4}$ and $\epsilon = \min\left\{\frac{\delta}{2}, \frac{1}{100}\right\}$. Then take $x = k^{10}$ and $y = k^{17/(3-2\alpha)}$ and we obtain the result. $\square$

Remark 3.2. *It can be seen from the proof that, the implied constant comes from the function $G_{f,g}(s)$. When we consider $\delta \geq \frac{1}{4}$, $\epsilon = \frac{1}{100}$. So the implied constant is absolute.*

Remark 3.3. *For the symmetric square L-function $L(s, \mathrm{Sym}^2 f)$, one can show that for $\alpha \geq \frac{1}{2} + \delta$,*

$$\sum_{f \in H_k} N(\alpha, T, \mathrm{Sym}^2 f) \ll_\delta T^2 (\log T) k^{22(1-\alpha)/(3-2\alpha)} (\log k)^{17}.$$

Furthermore, when $\frac{1}{2} + \delta$ is close to 1, the implied constant is absolute.

4 Proof of Theorem 1.4

Proof of Theorem 1.4: Fix $\epsilon > 0$. Set

$$\eta = \frac{(\log \log k)^2}{\log k} \in \left(\frac{3 \log \log k}{\log k}, \frac{1}{4} \right),$$

(This is true when k is large.) and we apply the Remark 1.3 and Corollary 1.2. This gives:

$$|H_k^-(\eta)| \ll k^{36\eta} (\log k)^{18} = o((\log k)^{37 \log \log k}) \tag{9}$$

and

$$|D_k^-(\eta)| \ll k^{36\eta} (\log k)^{26} = o((\log k)^{37 \log \log k}). \tag{10}$$

Then we claim: for $f \in H_k^+(\eta)$ and $(f, g) \in D_k^+(\eta)$, f and g are distinguishable. We will prove the claim later. Then we define

$$H_k(-, \eta) = \{ f \in H_k | \text{there exists } g \in H_k \text{ such that either } (f, g) \text{ or } (g, f) \text{ belongs to } D_k^-(\eta) \}$$

Then equation (10) implies that

$$|H_k(-, \eta)| \leq |D_k^-(\eta)| = o((\log k)^{37 \log \log k}). \tag{11}$$

Then set

$$H_k^* = H_k - (H_k^-(\eta) \cup H_k(-, \eta))$$

It can be seen that for distinct $f, g \in H_k^*$, we have $f \in H_k^+(\eta)$ and $(f, g) \in D_k^+(\eta)$. If we assume the claim, then f and g are distinguishable. Then by Equation (9) and (11), we have

$$\lim_{k \to \infty} \frac{|H_k^*|}{|H_k|} = 1.$$

So it suffices to prove the claim. The main idea is to compare two integrals, which comes from [GH93]. Let $f \in H^+(\eta)$ and $(f, g) \in D_k^+(\eta)$, $g \neq f$. For $\mathrm{Re}(s) > 1$, we define $\Lambda_{f \otimes f}(n)$ and $\Lambda_{f \otimes g}(n)$ by:

$$-\frac{L'(s, f \otimes f)}{L(s, f \otimes f)} = \sum_{n=1}^{\infty} \frac{\Lambda_{f \otimes f}(n)}{n^s}$$

and

$$-\frac{L'(s, f \otimes g)}{L(s, f \otimes g)} = \sum_{n=1}^{\infty} \frac{\Lambda_{f \otimes g}(n)}{n^s}.$$

Then set $a = 1 + \frac{1}{\log k}$ and $T = 50k^{\eta} = 50(\log k)^{\log\log k}$. For any $x \geq 1$, we consider

$$\mathrm{I} = \frac{1}{2\pi i}\int_{a-iT}^{a+iT}\left(\frac{x^{s-1/2}-x^{1/2-s}}{s-1/2}\right)^2\left(-\frac{L'(s,f\otimes f)}{L(s,f\otimes f)}\right)ds.$$

This gives:

$$\mathrm{I} = \sum_{n<x^2}\frac{\Lambda_{f\otimes f}(n)}{n^{1/2}}\log\left(\frac{x^2}{n}\right) + O\left(\frac{x^{2a-1}(\log k)}{T}\right) \tag{12}$$

since

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{y^{s-1/2}}{(s-1/2)^2}\,ds = \begin{cases} \log y & \text{if } y \geq 1 \\ 0 & \text{if } y \leq 1. \end{cases}$$

On the other hand, we consider the path which is the boundary of the rectangular:

$$\left\{z \in \mathbb{C}\,\middle|\, 1-\frac{3}{4}\eta \leq \mathrm{Re}(s) \leq a, |\,\mathrm{Im}(s)| \leq T\right\}.$$

Due to the choice of f, $L(s,f\otimes f) = \zeta(s)L(s,\mathrm{Sym}^2 f)$ has no zero in the rectangular: the rectangle is contained in the zero free region of Riemann zeta function when k large. Here we recall the well–known zero free region of Riemann zeta function proved by Vinogradov-Korobov [Vin58] is, for $s = \beta + it$ and

$$\beta \in \left(1 - \frac{c}{(\log t)^{2/3}(\log\log t)^{1/3}}, 1\right),$$

$\zeta(s) \neq 0$. On the other hand, since $f \in H_k(\eta)$, $L(s,\mathrm{Sym}^2 f)$ has no zeros in the rectangle. (Notice that the L-function is self-dual.) There is a simple pole of $L(s,f\otimes f) = \zeta(s)L(s,\mathrm{Sym}^2 f)$ at $s = 1$. Then by the residue theorem, we have

$$\begin{aligned}\mathrm{I} &= 4(x-2+x^{-1}) \\ &\quad + \frac{1}{2\pi i}\left(\int_{1-\frac{3}{4}\eta-i\infty}^{1-\frac{3}{4}\eta+i\infty} + \int_{1-\frac{3}{4}\eta+iT}^{a+iT} - \int_{1-\frac{3}{4}\eta-iT}^{a-iT}\right)\left(\frac{x^{s-1/2}-x^{1/2-s}}{s-1/2}\right)^2\left(-\frac{L'(s,f\otimes f)}{L(s,f\otimes f)}\right)ds \\ &= 4(x-2+x^{-1}) + \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3.\end{aligned}$$

Then by [IK04, Proposition 5.7], we have:

$$\mathrm{I}_2 + \mathrm{I}_3 \ll \frac{x^{2a-1}(\log k)^2}{T^2}. \tag{13}$$

For the term I_1, we have the trivial estimation:

$$\mathrm{I}_1 \ll x^{1-\frac{3}{2}\eta}(\log k)^2 \tag{14}$$

Then combine Equation (12), (13) and (14), and we obtain:

$$\sum_{n<x^2}\frac{\Lambda_{f\otimes f}(n)}{n^{1/2}}\log\left(\frac{x^2}{n}\right) = 4(x-2+x^{-1}) + O\left(\frac{x^{2a-1}(\log k)^2}{T}\right) + O(x^{1-\frac{3}{2}\eta}(\log k)^2). \tag{15}$$

Then we replace $-L'(s, f\otimes f)/L(s, f\otimes f)$ by $-L'(s, f\otimes g)/L(s, f\otimes g)$, that is, set

$$\mathrm{II} = \frac{1}{2\pi i}\int_{a-iT}^{a+iT}\left(\frac{x^{s-1/2}-x^{1/2-s}}{s-1/2}\right)^2\left(-\frac{L'(s,f\otimes g)}{L(s,f\otimes g)}\right)ds.$$

This will give:

$$\sum_{n<x^2}\frac{\Lambda_{f\otimes g}(n)}{n^{1/2}}\log\left(\frac{x^2}{n}\right) = O\left(\frac{x^{2a-1}(\log k)^2}{T}\right) + O(x^{1-\frac{3}{2}\eta}(\log k)^2). \tag{16}$$

since $L(s, f\otimes g)$ has neither zero nor poles in the rectangular.

Now suppose that $\Lambda_{f\otimes f}(n) = \Lambda_{f\otimes g}(n)$ for $n < x^2$. Then Equation (15) and (16) show:

$$0 = 4(x-2+x^{-1}) + O\left(\frac{x^{2a-1}(\log k)^2}{T}\right) + O(x^{1-\frac{3}{2}\eta}(\log k)^2).$$

For any $\epsilon > 0$, we set $x = k^{\epsilon/2}$. By the choice of c, η and T, this can not be true when k is large. So we can find $n \le x^2 = k^\epsilon$ such that $\Lambda_{f\otimes f}(n) \ne \Lambda_{f\otimes g}(n)$. Notice that $\Lambda_{f\otimes f}(n)$ and $\Lambda_{f\otimes g}(n)$ are supported on prime powers and totally determined by Satake parameters. Therefore, we can find $p \ll_\epsilon k^\epsilon$ such that $\lambda_f(p) \ne \lambda_g(p)$. □

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